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The coherent anomaly method and long-range one-dimensional Ising models

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Abstract. We present the results of the coherent anomaly method when applied to Ising models in one dimension with long-range interactions. This class of systems acts as an interesting and challenging test for the method because the critical exponents as well as the critical temperature, values of which are given by the method, depend on the rate of fall-off of the interactions. Thus one can see how accurately the method gives results which correctly reflect this dependency. The results obtained from this method are compared with results obtained by a variety of other methods.

1. Introduction

Since its recent introduction in 1986 by Suzuki [1] the coherent anomaly method (hereafter CAM) has been applied to an extremely wide range of critical phenomena including Ising spin systems [2-5], quantum spin systems [6], spin glasses [7], percolation [7, 8] and self-avoiding random walks [9]. The method allows one to calculate estimates for the critical properties, e.g. the critical temperature and critical exponents in the case of Ising models, relevant to the particular problem under study. To use the method one considers a sequence of self-consistent approximations. Suzuki has shown how to extract a feature he has named the coherent anomaly from this sequence of approximations. This is then used to obtain estimates for the relevant critical properties of the phenomena. The new estimates are a great improvement on those the individual approximations would give. For example many of the approximations used give 'classical' values for the critical exponents, e.g. in the case of the Ising models $\gamma = 1$, where the coherent anomaly method may give estimates of these critical exponents differing from the exact result by less than 1%, e.g. for the standard two-dimensional Ising model the method has been used to get $\gamma = 1.749$ [4].

The most extensive use of the method has involved the two- and three-dimensional Ising models with nearest-neighbour, ferromagnetic interactions [2-4] where the results, often extremely accurate, have been based on a number of different sequences of self-consistent approximations. We will consider one-dimensional Ising models with long-range ferromagnetic interactions. By long range we mean interactions falling off

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as $1/r^{\theta}$ where $1 < \theta \leq 2$ and the lattice spacing is of unit size. We have several reasons for choosing such systems.

(i) It is known that the critical exponents for these systems vary as a function of θ. Hence, rather than calculating one value of γ as with finite range two-dimensional systems and thus having only one comparison of the result, we have a range of values of γ dependent on θ and we make comparisons of our results over this entire range.

(ii) In past studies it has been seen that the larger the sequence of the self-consistent approximations one has available, the greater the accuracy of the coherent anomaly method estimates. For one-dimensional systems one can obtain a rather lengthy sequence of approximations with minimal amounts of computer time. Thus one can study how the accuracy of the estimates depends on the length of the sequence.

(iii) There has been extensive interest in these systems for a number of years now, particularly the $1/r^2$ case. This interest has resulted in a variety of rigorous results being established [10-12] as well as a variety of numerical results [13-16]. All the results show these systems to be delicate and therefore difficult to deal with.

For all the above reasons we feel the class of long-range, one-dimensional, ferromagnetic Ising models presents an interesting class of models to investigate using the coherent anomaly method and one where the method would be critically tested.

In section 2 we will very briefly outline the method and describe our sequence of approximations. Section 3 contains the numerical input, obtained from the approximations of the previous section, and the estimates the method gives for the critical temperature and critical exponents γ and β based on this input. The section also contains comparisons of our estimates with those values obtained by other methods. As stated earlier we believe this results in a new sensitive test of the method. Finally section 4 contains some concluding remarks.

2. Cluster mean-field and CAM

In this section we begin by describing our sequence of approximations which act as input for the coherent anomaly method. We then briefly outline Suzuki's method emphasising those aspects which are relevant to our particular analysis. For full details of the method we refer the reader to [1, 17, 18].

We start with a one-dimensional lattice of sites where on the ith site we have a spin variable $\sigma_i = \pm 1$. The interaction of the spins is governed by the Hamiltonian

$$\mathcal{H}(\{\sigma\}) = -\sum_{i<j} \frac{J}{|i-j|^{\theta}} \sigma_i \sigma_j \tag{1}$$

where $|i-j|$ represents the distance between sites i and j with the distance between adjacent sites set equal to 1. The thermal average of a spin is defined in the usual way

$$\langle \sigma_i \rangle = Z^{-1} \sum_{\{\sigma\}} \sigma_i \exp[-\beta \mathcal{H}(\{\sigma\})] \tag{2}$$

where Z is the partition function, the sum is over all configurations of spins and $\beta = 1/kT$.

Our sequence of self-consistent approximation consists of a sequence of what have been called cluster mean-field approximations. Here we treat all interactions among the members of the cluster exactly and we replace all interactions between a spin in the cluster and one outside the cluster with a mean-field interaction. For example the

Hamiltonian for the three-site cluster containing spins σ_1, σ_2, and σ_3 would be

$$\mathcal{H}(\sigma_1, \sigma_2, \sigma_3)$$
$$= -J[\sigma_1\sigma_2 + \sigma_2\sigma_3] - \frac{J}{2^\theta}\sigma_1\sigma_3 - Jm(\sigma_1+\sigma_3)\left[\sum_{n=1}^{\infty}\frac{1}{n^\theta} + \sum_{n=3}^{\infty}\frac{1}{n^\theta}\right]$$
$$- Jm\sigma_2\left[2\sum_{n=2}^{\infty}\frac{1}{n^\theta}\right] \tag{3}$$

where m represents the mean field. We then require in usual mean-field fashion that the thermal average of the middle spin of our cluster equal m, i.e. for the above case $\langle\sigma_2\rangle = m$. Then letting $m \to 0$ we have a cluster mean-field estimate of the critical temperature. We look at odd-site clusters from 1 to 17 sites thus generating a sequence of 9 clusters with only 17 sites in the largest cluster. In the case of the square lattice Ising system Suzuki *et al* [2] generates a sequence of 7 clusters with the 7th cluster containing 199 sites. Our restriction to a maximum of only 17 sites allows for a minimum of computation while still generating a reasonable sequence of input values.

In addition to the critical temperature we can look at the zero-field susceptibility χ_N where the subscript N will denote the cluster size. One finds

$$\chi_N = \bar{\chi}_N \frac{1}{\varepsilon} \qquad \varepsilon \equiv \frac{T - T_c(N)}{T_c(N)} \tag{4}$$

where $T_c(N)$ is the critical temperature of the N site cluster. As for the spontaneous magnetisation cluster, mean-field approximations give

$$m_N^* = \overline{m_N^*}|\varepsilon|^{1/2} \tag{5}$$

where the superscript * on m denotes that we have spontaneous magnetisation.

It is the values $T_c(N)$, $\bar{\chi}_N$ and $\overline{m_N^*}$ which Suzuki's method uses to calculate estimates to T_c, γ, and β where β here is the usual critical exponent not $\beta = 1/kT$. In particular, based on Fisher's scaling form of the correlation function, Suzuki shows

$$T_c(N) = T_c + C[\bar{\chi}_N]^{-1/(\gamma-1)} \tag{6}$$

where c is an appropriate constant. For a derivation see e.g. [2, section 3]. It is then easy to obtain

$$\gamma - 1 = \frac{\log(\bar{\chi}_N/\bar{\chi}_L)}{\log(\delta T_c(L)/\delta T_c(N))} \tag{7}$$

where $\delta T_c(N) = T_c(N) - T_c$ where T_c will denote the true critical temperature. In similar fashion Suzuki also finds

$$\tfrac{1}{2} - \beta = \frac{\log(\overline{m_N^*}/\overline{m_L^*})}{\log(\delta T_c(L)/\delta T_c(N))}. \tag{8}$$

We use (7) and (8) along with our cluster mean-field approximations to calculate values of T_c, γ and β.

3. Results and comparisons

In this section we present the results of our coherent anomaly method calculations. We have looked at values of θ in (1) from 1.1 to 2.0 in steps of 0.1. We present in

detail only the cases $\theta = 1.2$ and $\theta = 1.8$. These two values are representative of two general situations one encounters in these long-range interactions systems, as will be explained later. As stated in the previous section, clusters from 1 to 17 sites were considered generating a sequence of 9 values for each of $T_c(N)$, $\bar{\chi}_N$ and $\overline{m_N^*}$. These values constitute input into the coherent anomaly method and for $\theta = 1.2$ and $\theta = 1.8$ are given in table 1.

Any sequence of three clusters, and therefore three values of $T_c(N)$, $\bar{\chi}_N$, and $\overline{m_N^*}$, is enough using (7) and (8) to calculate T_c, γ, and β. However the best results in previous work have been obtained using the full sequence of values available and a least squares fit to obtain these results, (see e.g. [2, section 6]). We have analysed our data using both approaches.

We first present the results using a least squares fit. To try and investigate how to use the input most effectively in the coherent anomaly method we first look at a sequence involving only the 1, 3 and 5 site clusters, then the 1, 3, 5 and 7 site clusters, and continuing until we reach the sequence containing 1, 3, 5, . . . , 17 sites. In this way we illustrate the effect of increasing the length of the sequence. For $\theta = 1.2$ and 1.8 our results are presented in table 2.

As already mentioned only three values of $T_c(N)$, $\bar{\chi}_N$, and $\overline{m_N^*}$ are needed to use the method; thus it is useful to present the results obtained by looking at only sequences

Table 1. Data for the CAM estimates for the various cluster sizes.

number of sites	$\theta = 1.2$			$\theta = 1.8$		
	T_c	$\bar{\chi}_N$	$\overline{m_N^*}$	T_c	$\bar{\chi}_N$	$\overline{m_N^*}$
1	11.1832	0.0894	1.7320	3.7645	0.2656	1.7320
3	11.0170	0.0920	1.7526	3.3246	0.3336	1.8483
5	10.9641	0.0931	1.7619	3.1503	0.3797	1.9218
7	10.9373	0.0936	1.7675	3.0507	0.4160	1.9772
9	10.9211	0.0940	1.7714	2.9841	0.4465	2.0222
11	10.9100	0.0943	1.7743	2.9357	0.4732	2.0604
13	10.9019	0.0945	1.7766	2.8983	0.4970	2.0937
15	10.8957	0.0947	1.7785	2.8683	0.5186	2.1234
17	10.8909	0.0948	1.7800	2.8436	0.5385	2.1502

Table 2. CAM estimates for T_c, γ and β using increasing lengths of sequences. For T_c we have two estimates, one based on equation (7) and labelled $T_c(\gamma)$ and one based on equation (8) and labelled $T_c(\beta)$.

Sequence of clusters	$\theta = 1.2$				$\theta = 1.8$			
	$T_c(\gamma)$	γ	$T_c(\beta)$	β	$T_c(\gamma)$	γ	$T_c(\beta)$	β
1, 3, 5	10.478	1.107	10.728	0.4741	2.553	1.505	2.664	0.373
1, 3, 5, 7	10.496	1.104	10.255	0.4745	2.546	1.509	2.649	0.370
1, . . . , 9	10.514	1.101	10.252	0.4748	2.539	1.513	2.636	0.368
1, . . . , 11	10.527	1.099	10.249	0.4751	2.534	1.516	2.626	0.366
1, . . . , 13	10.538	1.097	10.247	0.4753	2.530	1.519	2.618	0.365
1, . . . , 15	10.547	1.095	10.245	0.4755	2.526	1.522	2.610	0.363
1, . . . , 17	10.555	1.093	10.243	0.4757	2.523	1.524	2.604	0.362

of three clusters but increasing the size of the clusters i.e., a 1, 3 and 5 site sequence, then a 3, 5 and 7 site sequence, etc. Such results are presented in table 3 again for $\theta = 1.2$ and $\theta = 1.8$.

Unlike the case of the nearest-neighbour, two-dimensional Ising model our system does not have exact results for comparison. Even for the nearest-neighbour, three-dimensional Ising model, one has a tremendous amount of previous numerical work resulting in very precise estimates of the critical temperature and critical-point exponents. Nevertheless for one-dimensional, long-range interaction systems one does have some estimates of the quantities under investigation for the entire range of θ values for which there is a phase transition i.e. $1 < \theta < 2$. Extensive numerical calculations involving finite chains and an extrapolation based on Padé approximates were performed by Nagle and Bonner [15]. Using a renormalisation group approach Fisher, Ma and Nickel [13] calculate estimates of γ. Also the work of Doman [19] and Silves, Pires and Ferreira [20] contain estimates of T_c.

Table 3. CAM estimates for T_c, γ and β using sequences involving only three clusters. For T_c we have two estimates, one based on equation (7) and labelled $T_c(\gamma)$ and one based on equation (8) and labelled $T_c(\beta)$.

Sequence of clusters	$\theta = 1.2$				$\theta = 1.8$			
	$T_c(\gamma)$	γ	$T_c(\beta)$	β	$T_c(\gamma)$	γ	$T_c(\beta)$	β
1, 3, 5	10.478	1.107	10.728	0.474	2.553	1.505	2.664	0.373
3, 5, 7	10.574	1.087	10.756	0.477	2.524	1.527	2.608	0.360
5, 7, 9	10.673	1.064	10.770	0.479	2.506	1.543	2.575	0.351
7, 9, 11	10.637	1.073	10.773	0.479	2.494	1.556	2.553	0.343
9, 11, 13	10.709	1.055	10.776	0.479	2.485	1.567	2.538	0.337
11, 13, 15	10.724	1.050	10.786	0.481	2.479	1.575	2.527	0.333
13, 15, 17	10.759	1.040	10.791	0.482	2.473	1.584	2.517	0.328

We note that in [13] it is shown that based on renormalisation group calculations for $1 < \theta < 1.5$ one has $\gamma = 1$, i.e., the critical point exponent takes its classical value. It is for this reason that we have chosen $\theta = 1.2$ and $\theta = 1.8$ as two representative values since one falls near the middle of the θ-interval where γ takes on its classical value and one near the middle of the region where γ takes on its non-classical values.

For $\theta = 1.2$ table 2 shows that as the length of the sequence of clusters increases γ and β both approach their classical values in a very systematic fashion. Similarly table 3 shows that considering only sequences of three clusters the values for γ and β again approach their classical values in a systematic way as the size of the three clusters being considered increases. Comparing the two tables one sees that the sequence of the 13, 15 and 17 site clusters results in the best estimate for γ and β where we assume the classical values are the correct values. Therefore, we see we need only consider three clusters simply considering the largest available. This however does not always hold. We have analysed the data of Suzuki *et al* [2] for the two-dimensional nearest-neighbour Ising model involving seven different clusters ranging in size from 1 site to 145 sites. Using the full sequence of seven clusters Suzuki *et al* obtained $\gamma = 1.7$ (+0.05) and $T_c = 2.24$ (0.036). Our results taking three clusters at a time are given in table 4. One does not see the systematic approach to the true values that we see for the long-range one-dimensional Ising systems.

Table 4. CAM estimates for T_c and γ of the nearest-neihbour two-dimensional Ising model on the square lattice using the published data of Suzuki, Katori and Hu [2].

Sequence of clusters	T_c	γ
1, 9, 21	2.0154	1.8221
9, 21, 45	2.4102	1.4958
21, 45, 69	2.0475	1.9315
45, 69, 97	2.2958	1.5834
69, 97, 145	2.3659	1.4693

Table 5. Comparisons of T_c, γ and β CAM estimates with previous work. All CAM estimates are based on a three cluster sequence involving 13, 15 and 17 site clusters.

θ of $1/r^\theta$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
CAM estimate for $T_c(\beta)$	20.959	10.791	7.298	5.492	4.363	3.577	2.987	2.517	2.116	1.750
CAM estimate for $T_c(\gamma)$	20.908	10.758	7.244	5.430	4.283	3.503	2.927	2.473	2.114	1.807
Nagle-Bonner [15] estimate for T_c		10.997	7.302	5.478	4.334	3.532	2.917	2.424	2.003	1.633
CAM estimate for γ	1.014	1.040	1.097	1.167	1.264	1.360	1.463	1.584	1.701	1.846
Nagle-Bonner [15] estimate for γ		1.0	1.01	1.06	1.11	1.19	1.29	1.50	1.78	2.2
Fisher *et al* [13] estimate for γ	1.000	1.000	1.000	1.000	1.000	1.176	1.440	1.790	2.226	2.750
CAM estimate for β	0.495	0.482	0.460	0.435	0.408	0.381	0.354	0.328	0.299	0.265
Nagle-Bonner [15] estimate for β		0.5	0.48	0.45	0.39	0.33	0.26	0.18	0.10	0.0

We also note that, in general, the estimates of T_c are also increasing in both tables 2 and 3 for $\theta = 1.2$. One does not have accurate estimates of T_c so one cannot say unequivocally that T_c is approaching the true value. However, based on the fact that the values for both γ and β were approaching the correct values this gives us reason to believe this is true for T_c as well. Furthermore we believe the same to be true for the $\theta = 1.8$ case where again we see for T_c γ and β change in a very systematic fashion. We note that the changes which occur in tables 2 and 3 for $\theta = 1.2$ are the reverse of those that occur for $\theta = 1.8$. Hence in one case the estimated values of the coherent anomaly method approach the true value from the low side while in the other case the estimated values approach from the opposite side.

Based on the results of our detailed analysis for $\theta = 1.2$ and 1.8 we see that we need only present the estimates of T_c, γ and β being based on the sequence of 13, 15 and 17 site clusters for the remaining values that we have studied. In table 5 we list our estimates for γ, β and T_c along with, for comparison, the estimates of Nagle and Bonner [15] for the same three quantities and, for γ only, the results of Fisher *et al* [13]. For $\theta = 2.0$ we have additional work to compare our results to. In this case one has T_c estimates by Bhattacharjee *et al* [14] of 1.587 and by Matvienko [16] of 1.522.

4. Conclusions

From the above sections we see that the coherent anomaly method results properly reflect the dependence of γ, β and T_c on θ. In addition the method in all cases studied

here gives systematically improving results as the size of the clusters used in the method increases. This has been shown for what is an entire class of systems due to the fact that θ acts as a variable upon which all quantities depend. This is to be contrasted with a single system like the nearest-neighbour Ising model on a square lattice for which a single value for T_c, γ and β exists. This study further indicates the general applicability and versatility of the method when applied to spin systems.

It should however be kept in mind that the accuracy of the results are not significantly better than results obtained by other methods. In fact for small θ the renormalisation group results of Fisher, Ma and Nickel [13] are clearly superior. The results are similar to those obtained by Nagle and Bonner [15] and in terms of the work necessary to obtain them it would appear to be similar in amount.

We want to mention that there is another method for estimating critical properties of spin systems, due to Indekeu, Maritan and Stella [21], which has some general features similar to the coherent anomaly method. In particular the method uses mean-field results based on two cluster sizes along with a normalisation group strategy to obtain estimates of the critical properties. Unfortunately the method has not been used to investigate the systems considered in this paper.

Finally we should point out that we have not made an attempt to achieve the best possible results by use of large amounts of computer time on a very powerful computer. Rather, as stated earlier, the attempt was to investigate the general properties of the CAM as it relates to long-range interaction models. All computations for $T_c(N)$, $\overline{m_N^*}$, T_c and β were done on a personal computer. The remaining calculations could also have easily been done on a personal computer but were not simply because we initially started on a mainframe.

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References

[1] Suzuki M 1986 *Phys. Lett.* **116A** 375
[2] Suzuki M, Katori M and Hu X 1987 *J. Phys. Soc. Japan* **56** 3092
[3] Katori M and Suzuki M 1987 *J. Phys. Soc. Japan* **56** 3113
[4] Hu X, Katori M and Suzuki M 1987 *J. Phys. Soc. Japan* **56** 3865
[5] Monroe J 1988 *Phys. Lett.* **121A** 427
[6] Ito M and Suzuki M 1988 *Int. J. Mod. Phys.* **2** 1
[7] Suzuki M 1988 *Phys. Lett.* **127A** 410
[8] Takayasu M and Takayasu H 1988 *Phys. Lett.* **128A** 45
[9] Hu X and Suzuki M 1988 *Physica* **150A** 310
[10] Dyson F J 1971 *Commun. Math. Phys.* **21** 269
[11] Frohlich J and Spencer T 1982 *Commun. Math. Phys.* **84** 87
[12] Aizenman M, Chayes J T, Chayes L and Newman C M 1988 *J. Stat. Phys.* **50** 1
[13] Fisher M E, Ma S and Nickel B G 1972 *Phys. Rev. Lett.* **29** 917
[14] Bhattacharjee, Chakravarty S, Richardson J L and Scalapino D J 1981 *Phys. Rev.* B **24** 3862
[15] Nagle J F and Bonner J C 1970 *J. Phys. C: Solid State Phys.* **3** 352
[16] Matvienko G V 1985 *Teor. Mat. Fiz.* **63** 465

[17] Suzuki M 1985 *Proc. Int. Symp. on Quantum Field Theory in honor of H Umezawa* (Amsterdam: North-Holland)
[18] Suzuki M 1986 *J. Phys. Soc. Japan* **55** 4205
[19] Doman G S 1981 *Phys. Status Solidi* b **103** K169
[20] Silva N P, Pires A S T and Ferreira J F R, 1980 *Phys. Status Solidi* b **99** K23
[21] Indekeu J O, Maritan A and Stella A L 1982 *J. Phys. A: Math. Gen.* **15** L291